Similarly, this same point P is the centre of symmetry of parallelogram $BCFG$, and so on around the octagon. Thus, P is the centre of symmetry of the octagon.

8. Let $M = \{x^2 + x \mid x \text{ is a positive integer}\}$. For each integer $k \geq 2$ prove that there exist a_1, a_2, $\ldots$, a_k, b_k in M such that $a_1 + a_2 + \cdots + a_k = b_k$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

The proof is by induction on k.

Note that $x^2 + x = x(x + 1)$, so each element of M is even.

Since $12 + 30 = 42$ and $12 = 3 \cdot 4$, $30 = 5 \cdot 6$, $42 = 6 \cdot 7 \in M$, the desired result holds for $k = 2$.

Now the inductive step. Suppose that for some $k \geq 2$ we have $a_1 + a_2 + \cdots + a_k = b_k$, where $a_1, a_2, \ldots, a_k$, $b_k \in M$. Since b_k is even, we have $b_k = 2c$ for some positive integer c. Moreover, $b_k \geq a_1 + a_2 \geq 4$, so $c \geq 2$. Let $a_{k+1} = (c - 1)c \in M$. Then

$$a_1 + a_2 + \cdots + a_k + a_{k+1} \;=\; 2c + (c - 1)c \;=\; c(c + 1) \in M,$$

and the induction is complete.

Next we turn to solutions from our readers to problems of the Republic of Moldova Mathematical Olympiad Second and Third Team Selection Tests given at [2009 : 378–379].

3. Let a, b, c be the side lengths of a triangle and let s be the semiperimeter. Prove that

$$a\sqrt{\frac{(s-b)(s-c)}{bc}} \;+\; b\sqrt{\frac{(s-c)(s-a)}{ac}} \;+\; c\sqrt{\frac{(s-a)(s-b)}{ab}} \;\geq\; s\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $x := s - a, y := s - b, z := s - c$ then $x, y, z > 0$, $a = y + z$, $b = z + x$, $c = x + y$, $s = x + y + z$ and the original inequality becomes

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \geq x + y + z,$$

where $x, y, z > 0.$

Since

$$\sum_{cyc} (y+z)\sqrt{\frac{yz}{(x+y)(z+x)}} \;=\; \sum_{cyc} \frac{(y+z)\sqrt{yz\,(x+y)(z+x)}}{(x+y)(z+x)}$$

and by Cauchy and AM-GM Inequalities

$$\begin{aligned}(y+z)\sqrt{yz(x+y)(z+x)} &\geq (y+z)\sqrt{yz\left(x+\sqrt{yz}\right)^2}\\ &= (y+z)\sqrt{yz}\left(x+\sqrt{yz}\right)\\ &= x(y+z)\sqrt{yz}+(y+z)\,yz\\ &\geq 2x\sqrt{yz}\sqrt{yz}+(x+y)yz\\ &= 2xyz+(y+z)\,yz\\ &= yz\left((x+y)+(x+z)\right)\end{aligned}$$

then

$$\begin{aligned}\sum_{cyc}\frac{(y+z)\sqrt{yz(x+y)(z+x)}}{(x+y)(z+x)} &\geq \sum_{cyc}\frac{yz\left((x+y)+(x+z)\right)}{(x+y)(z+x)}\\ &= \sum_{cyc}\left(\frac{yz}{z+x}+\frac{yz}{x+y}\right)\\ &= \sum_{cyc}\frac{yz}{z+x}+\sum_{cyc}\frac{yz}{x+y}\\ &= \sum_{cyc}\frac{zx}{x+y}+\sum_{cyc}\frac{yz}{x+y}=\sum_{cyc}\frac{zx+yz}{x+y}\\ &= \sum_{cyc}\frac{z(x+y)}{x+y}=x+y+z.\end{aligned}$$

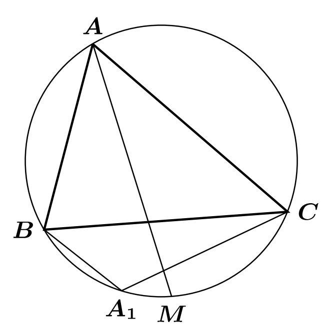
5. The point P is in the interior of triangle ABC. The rays AP, BP, and CP cut the circumcircle of the triangle at the points A_1, B_1, and C_1, respectively. Prove that the sum of the areas of the triangles A_1BC, B_1AC, and C_1AB does not exceed $s(R-r)$, where s, R, and r are the semiperimeter, the circumradius, and the inradius of triangle ABC, respectively.

Solution by Titu Zvonaru, Cománești, Romania.

We will prove the statement of the problem for the points A_1, B_1, C_1 such that A_1 belongs to arc BC which does not contain point A, and similarly for B_1 and C_1.

Let $[XYZ]$ be the area of $\triangle XYZ$. We denote $a=BC$, $b=CA$, $c=AB$. Let M be the midpoint of arc BC which contains the point A_1 (which does not contain the point A). It is easy to see that

$$[A_1BC]\ \leq\ [MBC]. \qquad (1)$$



We have

$$\angle MBC=\angle MCB=\angle MAC=\frac{A}{2}.$$